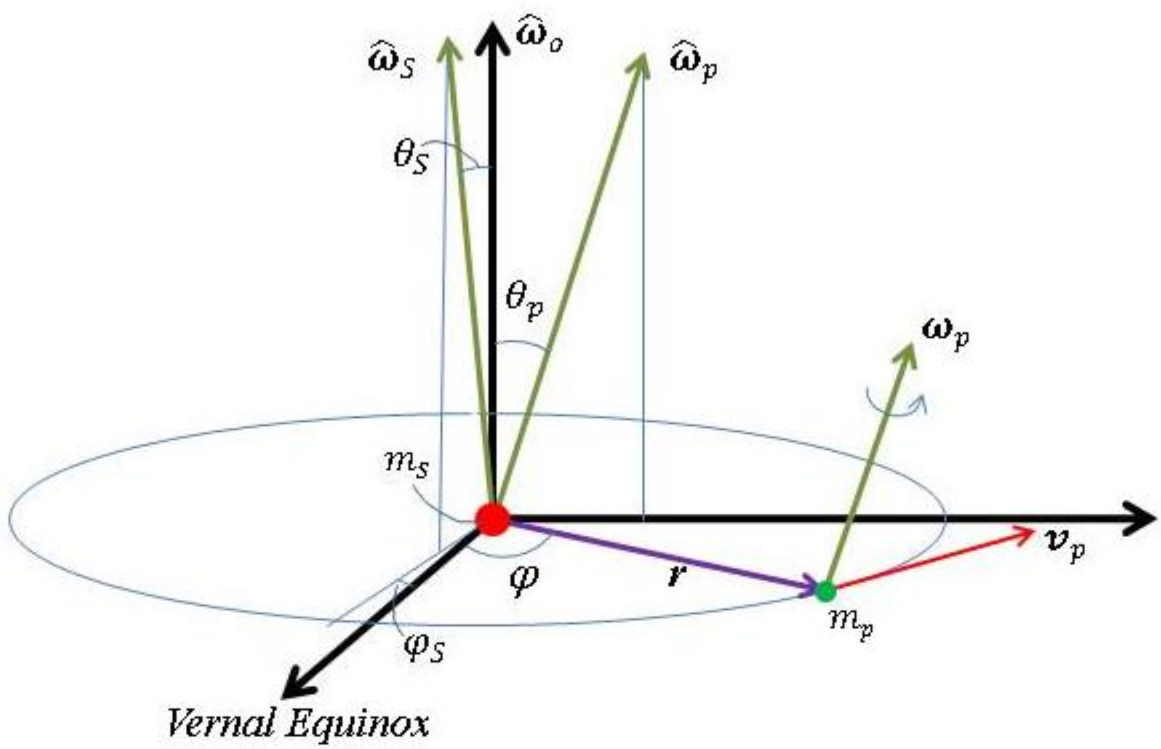
Mathematical Details for the article called "Universal force-motion equation and solar system implementation"

Note: Some formulas in red below are labelled with some letters. Those letters show the column name in the excel file created for scientists to study on the subject. Scientists can require the excel file by an e-mail notice directly from ayalcin@eser.com.



Reference Figure

$\hat{\boldsymbol{\omega}}_o(0,0)$	: Planet orbit normal unit vector (z-axis)
$x - y$ Plane	: Planet orbit plane
x-axis	: Vernal equinox
$\hat{\boldsymbol{\omega}}_S(\theta_S, \varphi_S)$	**:** Sun rotation axis unit vector
$\hat{\boldsymbol{\omega}}_p(\theta_p, \pi/2)$	**:** Planet rotation axis unit vector
$\boldsymbol{r}$	**:** Planet position vector

In Cartesian coordinates:

$\hat{\boldsymbol{\omega}}_o = \boldsymbol{k}$,

$\hat{\boldsymbol{\omega}}_p = \left(\sin\theta_p\,\boldsymbol{j} + \cos\theta_p\,\boldsymbol{k}\right)$ and

$\hat{\boldsymbol{\omega}}_S = (\sin\theta_S\cos\varphi_S\,\boldsymbol{i} + \sin\theta_S\sin\varphi_S\,\boldsymbol{j} + \cos\theta_S\,\boldsymbol{k})$

Considering, $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ in spherical coordinates as[1]:

$\boldsymbol{i} = \left(\sin\theta\cos\varphi\,\hat{\boldsymbol{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}\right)$

$\boldsymbol{j} = \left(\sin\theta\sin\varphi\,\hat{\boldsymbol{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}\right)$

$\boldsymbol{k} = \left(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\;\hat{\boldsymbol{\theta}}\right)$

(Where r radial distance ($\hat{\boldsymbol{r}}$), θ polar angle $\left(\hat{\boldsymbol{\theta}}\right)$, φ azimuth angle ($\hat{\boldsymbol{\varphi}}$))

In spherical coordinates:

$\hat{\boldsymbol{\omega}}_o = \boldsymbol{k} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\;\hat{\boldsymbol{\theta}}$,

$\hat{\boldsymbol{\omega}}_p = \left(\sin\theta_p\sin\theta\sin\varphi + \cos\theta_p\cos\theta\right)\hat{\boldsymbol{r}} + \left(\sin\theta_p\cos\theta\sin\varphi - \cos\theta_p\sin\theta\right)\hat{\boldsymbol{\theta}} + \sin\theta_p\cos\varphi\,\hat{\boldsymbol{\varphi}}$

$$\hat{\boldsymbol{\omega}}_S = [\sin\theta_S\sin\theta\cos(\varphi-\varphi_S) + \cos\theta_S\cos\theta]\hat{\boldsymbol{r}} + [\sin\theta_S\cos\theta\cos(\varphi-\varphi_S) - \cos\theta_S\sin\theta]\,\hat{\boldsymbol{\theta}} - \sin\theta_S\sin(\varphi-\varphi_S)\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}} = \begin{bmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\varphi}} \\ [\sin\theta_S\sin\theta\cos(\varphi-\varphi_S) + \cos\theta_S\cos\theta] & [\sin\theta_S\cos\theta\cos(\varphi-\varphi_S) - \cos\theta_S\sin\theta] & -\sin\theta_S\sin(\varphi-\varphi_S) \\ 1 & 0 & 0 \end{bmatrix}$$

$$\hat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}} = -\sin\theta_S\sin(\varphi-\varphi_S)\hat{\boldsymbol{\theta}} - [\sin\theta_S\cos\theta\cos(\varphi-\varphi_S) - \cos\theta_S\sin\theta]\,\hat{\boldsymbol{\varphi}}$$

$$|\hat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}| = \sqrt{\sin^2\theta_S\sin^2(\varphi-\varphi_S) + (\sin\theta_S\cos\theta\cos(\varphi-\varphi_S) - \cos\theta_S\sin\theta)^2}$$

On the orbital plane:

$$|\hat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}| = \sqrt{\sin^2\theta_S\sin^2(\varphi-\varphi_S) + \cos^2\theta_S}$$

$\hat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}$ for the point where the Sun is**(**$\varphi = \varphi + \pi, \theta = \pi - \theta$, i.e. $\sin(\varphi+\pi) = -\sin\varphi$,

$\cos(\varphi+\pi) = -\cos\varphi, \sin(\pi-\theta) = \sin\theta, \cos(\pi-\theta) = -\cos\theta$)

$$\hat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}} = \begin{bmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\varphi}} \\ \left(-\sin\theta_p\sin\theta\sin\varphi + \cos\theta_p\cos\theta\right) & \left(\sin\theta_p\cos\theta\sin\varphi - \cos\theta_p\sin\theta\right) & -\sin\theta_p\cos\varphi \\ 1 & 0 & 0 \end{bmatrix}$$

$$\left(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}\right) = -\sin\theta_p\cos\varphi\,\hat{\boldsymbol{\theta}} - \left(\sin\theta_p\cos\theta\sin\varphi - \cos\theta_p\sin\theta\right)\hat{\boldsymbol{\varphi}}$$

$$(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \widehat{\boldsymbol{r}}) \cdot (\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \widehat{\boldsymbol{r}})$$
$$= \sin\theta_S \sin(\varphi - \varphi_S) \sin\theta_p \cos\varphi$$
$$+ [\sin\theta_S \cos\theta \cos(\varphi - \varphi_S) - \cos\theta_S \sin\theta]\big(\sin\theta_p \cos\theta \sin\varphi - \cos\theta_p \sin\theta\big)$$

$$(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \widehat{\boldsymbol{r}}) \cdot (\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \widehat{\boldsymbol{r}})$$
$$= \sin\theta_p \sin\theta_S \sin(\varphi - \varphi_S) \cos\varphi + \sin\theta_S \cos^2\theta \cos(\varphi - \varphi_S) \sin\theta_p \sin\varphi$$
$$- \cos\theta_S \sin\theta \sin\theta_p \cos\theta \sin\varphi - \cos\theta_p \sin\theta \sin\theta_S \cos\theta \cos(\varphi - \varphi_S)$$
$$+ \cos\theta_S \cos\theta_p \sin^2\theta$$

$$(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \widehat{\boldsymbol{r}}) \cdot (\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \widehat{\boldsymbol{r}})$$
$$= \sin\theta_S \sin\theta_p \sin(\varphi - \varphi_S) \cos\varphi + \sin\theta_S \sin\theta_p \cos^2\theta \cos(\varphi - \varphi_S) \sin\varphi$$
$$- \big[\sin\theta_p \cos\theta_S \sin\varphi + \sin\theta_S \cos\theta_p \cos(\varphi - \varphi_S)\big] \sin\theta \cos\theta + \cos\theta_p \sin^2\theta \cos\theta_S$$

On the orbital plane:

AF $$(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \widehat{\boldsymbol{r}}) \cdot (\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \widehat{\boldsymbol{r}}) = \sin\theta_S \sin(\varphi - \varphi_S) \sin\theta_p \cos\varphi + \cos\theta_S \cos\theta_p$$

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The area of the ellipse planet sweeps during one period(T) where a and b are semi major and semi minor axes respectively:

$$S = \pi ab$$

The area swept in unit time:

$$s = \frac{\pi ab}{T}$$

Planet velocity in spherical coordinates:

$$\boldsymbol{v}_{\boldsymbol{p}} = \frac{d\boldsymbol{r}}{dt} = \dot{r}\widehat{\boldsymbol{r}} + r\dot{\theta}\widehat{\boldsymbol{\theta}} + r\dot{\varphi}\sin\theta\,\widehat{\boldsymbol{\varphi}}$$

The area the planet swept in unit time:

$$s = \frac{1}{2}|\boldsymbol{r} \times \boldsymbol{v}| = \frac{1}{2}r|\widehat{\boldsymbol{r}} \times \boldsymbol{v}|$$

$$s = \frac{1}{2}r|\widehat{\boldsymbol{r}} \times \boldsymbol{v}| = \frac{1}{2}r\left\|\begin{matrix} \widehat{\boldsymbol{r}} & \widehat{\boldsymbol{\theta}} & \widehat{\boldsymbol{\varphi}} \\ 1 & 0 & 0 \\ \dot{r} & r\dot{\theta} & r\dot{\varphi}\sin\theta \end{matrix}\right\|$$

$$s = \frac{1}{2}r\left|-r\dot{\varphi}\sin\theta\,\widehat{\boldsymbol{\theta}} + r\dot{\theta}\widehat{\boldsymbol{\varphi}}\right| = \frac{1}{2}r^2\sqrt{\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2}$$

$$\frac{\pi ab}{T} = \frac{1}{2}r^2\sqrt{\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2}$$

$$\sqrt{\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2} = \frac{2\pi ab}{Tr^2}$$

With: $\omega_o = \frac{2\pi}{T}$

$$\dot{\varphi}^2 = \frac{\omega_o^2 a^2 b^2}{r^4 \sin^2\theta} - \frac{\dot{\theta}^2}{\sin^2\theta}$$

$$\dot{\varphi} = \sqrt{\frac{\omega_0^2 a^2 b^2}{r^4 \sin^2\theta} - \frac{\dot{\theta}^2}{\sin^2\theta}}$$

On the orbit plane since $\theta = constant = 90°, \dot{\theta} = 0, \sin 90° = 1$:

T

$$\dot{\varphi} = \frac{\omega_o ab}{r^2}$$

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Position vector of the planet[2] (ε: eccentricity, Φ: true anomaly):

$$\boldsymbol{r} = a\frac{1-\varepsilon^2}{1+\varepsilon\cos\Phi}\hat{\boldsymbol{r}}$$

For true anomaly in general:

Radial unit vector:

$$\hat{\boldsymbol{r}} = (\sin\theta\cos\varphi\ \boldsymbol{i} + \sin\theta\sin\varphi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k})$$

If the unit vector pointing the perihelion $\hat{\boldsymbol{u}}_{\boldsymbol{PH}}$ $(\theta_{PH}, \varphi_{PH})$**:**

$$\hat{\boldsymbol{u}}_{\boldsymbol{PH}} = (\sin\theta_{PH}\cos\varphi_{PH}\ \boldsymbol{i} + \sin\theta_{PH}\sin\varphi_{PH}\,\boldsymbol{j} + \cos\theta_{PH}\,\boldsymbol{k})$$

The perihelion is the angle between $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{u}}_{\boldsymbol{PH}}$ then:

$$\cos\Phi = \hat{\boldsymbol{r}}.\hat{\boldsymbol{u}}_{\boldsymbol{PH}} = \sin\theta\cos\varphi\sin\theta_{PH}\cos\varphi_{PH} + \sin\theta\sin\varphi\sin\theta_{PH}\sin\varphi_{PH} + \cos\theta\cos\theta_{PH}$$

$$\cos\Phi = \sin\theta\sin\theta_{PH}\,(\cos\varphi\cos\varphi_{PH} + \sin\varphi\sin\varphi_{PH}) + \cos\theta\cos\theta_{PH}$$

$$\cos\Phi = [\sin\theta\sin\theta_{PH}\cos(\varphi - \varphi_{PH}) + \cos\theta\cos\theta_{PH}]$$

On the orbit plane: $\theta_{PH} = \pi/2$ then:

$$\Phi = (\varphi - \varphi_{PH})$$

$$\boldsymbol{r} = a\frac{1-\varepsilon^2}{1+\varepsilon\cos(\varphi - \varphi_{PH})}\hat{\boldsymbol{r}}$$

S

$$r = a\frac{1-\varepsilon^2}{1+\varepsilon\cos(\varphi - \varphi_{PH})}$$

U $$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{\varepsilon r\dot{\varphi}\sin(\varphi - \varphi_{PH})}{1 + \varepsilon\cos(\varphi - \varphi_{PH})}$$

$$\boldsymbol{v_p} = \frac{d\boldsymbol{r}}{dt} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}$$

On the orbit plane:

AB $$|\boldsymbol{v_p}| = |\dot{\boldsymbol{r}}| = \sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}$$

$$\dot{\varphi} = \frac{\omega_o ab}{r^2}$$

$$\ddot{\varphi} = \frac{d\dot{\varphi}}{dt} = \frac{d\dot{\varphi}}{dr}\frac{dr}{dt} = -2r\frac{\omega_o ab}{r^4}\dot{r}$$

V $$\ddot{\varphi} = -2\frac{\dot{r}\dot{\varphi}}{r}$$

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$$\dot{r} = \frac{\varepsilon r\dot{\varphi}\sin(\varphi - \varphi_{PH})}{1 + \varepsilon\cos(\varphi - \varphi_{PH})}$$

W $$B = 1 + \varepsilon\cos(\varphi - \varphi_{PH})$$

X $$\dot{B} = -\varepsilon\dot{\varphi}\sin(\varphi - \varphi_{PH})$$

Y $$A = \dot{\varphi}\sin(\varphi - \varphi_{PH})$$

Z $$\dot{A} = [\dot{\varphi}^2\cos(\varphi - \varphi_{PH}) + \ddot{\varphi}\sin(\varphi - \varphi_{PH})]$$

$$\dot{r} = \varepsilon r\frac{A}{B}$$

$$\ddot{r} = \varepsilon\frac{d}{dt}r\frac{A}{B}$$

$$\ddot{r} = \varepsilon\left(\frac{A}{B}\frac{d}{dt}r + r\frac{1}{B}\frac{d}{dt}A + rA\frac{d}{dt}\frac{1}{B}\right)$$

$$\ddot{r} = \varepsilon\left(\frac{A}{B}\dot{r} + \frac{r}{B}\dot{A} - \frac{\varepsilon}{\varepsilon}\frac{rA}{B}\frac{\dot{B}}{B}\right)$$

$$\ddot{r} = \varepsilon\left(\frac{A}{B}\dot{r} + \frac{r}{B}\dot{A} - \dot{r}\frac{\dot{B}}{\varepsilon B}\right)$$

AA

$$\ddot{r} = \frac{\varepsilon}{B}\left[\left(A - \frac{\dot{B}}{\varepsilon}\right)\dot{r} + r\dot{A}\right]$$

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$$\dot{\varphi} = \frac{\omega_o ab}{r^2}$$

$$\frac{\partial \dot{\varphi}}{\partial \varphi} = 0$$

$$\frac{\partial}{\partial r}\dot{\varphi} = \frac{\partial}{\partial r}\frac{\omega_o ab}{r^2}$$

$$\frac{\partial \dot{\varphi}}{\partial r} = -\frac{2r\omega_o ab}{r^4}$$

$$\frac{\partial \dot{\varphi}}{\partial r} = -\frac{2\omega_o ab}{rr^2}$$

$$\frac{\partial \dot{\varphi}}{\partial r} = -2\frac{\dot{\varphi}}{r}$$

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$$\dot{r} = \frac{\varepsilon r\dot{\varphi}\sin(\varphi - \varphi_{PH})}{1 + \varepsilon\cos(\varphi - \varphi_{PH})}$$

$$\frac{\partial \dot{r}}{\partial r} = \frac{\varepsilon\dot{\varphi}\sin(\varphi - \varphi_{PH})}{1 + \varepsilon\cos(\varphi - \varphi_{PH})} + \frac{\varepsilon r\sin(\varphi - \varphi_{PH})}{1 + \varepsilon\cos(\varphi - \varphi_{PH})}\frac{\partial \dot{\varphi}}{\partial r}$$

$$\frac{\partial \dot{r}}{\partial r} = \frac{\dot{r}}{r} + \frac{\dot{r}}{\dot{\varphi}}\frac{\partial \dot{\varphi}}{\partial r}$$

$$\frac{\partial \dot{r}}{\partial r} = -\frac{\dot{r}}{r}$$

$$\frac{\partial \dot{r}}{\partial \varphi} = \frac{\partial}{\partial \varphi}\frac{\varepsilon r\dot{\varphi}\sin(\varphi - \varphi_{PH})}{1 + \varepsilon\cos(\varphi - \varphi_{PH})}$$

$$\frac{\partial \dot{r}}{\partial \varphi} = \varepsilon r\dot{\varphi}\frac{\partial}{\partial \varphi}\frac{\sin(\varphi - \varphi_{PH})}{1 + \varepsilon\cos(\varphi - \varphi_{PH})} + \frac{\varepsilon r\sin(\varphi - \varphi_{PH})}{1 + \varepsilon\cos(\varphi - \varphi_{PH})}\frac{\partial \dot{\varphi}}{\partial \varphi}$$

$$\frac{\partial \dot{r}}{\partial \varphi} = \varepsilon r\dot{\varphi}\frac{\partial}{\partial \varphi}\frac{\sin(\varphi - \varphi_{PH})}{1 + \varepsilon\cos(\varphi - \varphi_{PH})}$$

$$\frac{\partial \dot{r}}{\partial \varphi} = \varepsilon r\dot{\varphi}\frac{[1 + \varepsilon\cos(\varphi - \varphi_{PH})]\cos(\varphi - \varphi_{PH}) + \varepsilon\sin^2(\varphi - \varphi_{PH})}{[1 + \varepsilon\cos(\varphi - \varphi_{PH})]^2}$$

$$\frac{\partial \dot{r}}{\partial \varphi} = \varepsilon r\dot{\varphi}\left\{\frac{\cos(\varphi - \varphi_{PH})}{[1 + \varepsilon\cos(\varphi - \varphi_{PH})]} + \frac{\varepsilon\sin^2(\varphi - \varphi_{PH})}{[1 + \varepsilon\cos(\varphi - \varphi_{PH})]^2}\right\}$$

$$\frac{\partial \dot{r}}{\partial \varphi} = \left\{ \frac{\cos(\varphi - \varphi_{PH})}{\sin(\varphi - \varphi_{PH})} \frac{\varepsilon r \dot{\varphi} \sin(\varphi - \varphi_{PH})}{[1 + \varepsilon \cos(\varphi - \varphi_{PH})]} + \frac{\varepsilon^2 r \dot{\varphi} \sin^2(\varphi - \varphi_{PH})}{[1 + \varepsilon \cos(\varphi - \varphi_{PH})]^2} \right\}$$

$$\frac{\partial \dot{r}}{\partial \varphi} = \left\{ \frac{\cos(\varphi - \varphi_{PH})}{\sin(\varphi - \varphi_{PH})} \dot{r} + \frac{1}{r \dot{\varphi}} \frac{\varepsilon^2 r^2 \dot{\varphi}^2 \sin^2(\varphi - \varphi_{PH})}{[1 + \varepsilon \cos(\varphi - \varphi_{PH})]^2} \right\}$$

$$\frac{\partial \dot{r}}{\partial \varphi} = \left\{ \frac{\cos(\varphi - \varphi_{PH})}{\sin(\varphi - \varphi_{PH})} \dot{r} + \frac{\dot{r}^2}{r \dot{\varphi}} \right\}$$

$$\frac{\partial \dot{r}}{\partial \varphi} = \dot{r} \left\{ \frac{1}{\tan(\varphi - \varphi_{PH})} + \frac{\dot{r}}{r \dot{\varphi}} \right\}$$

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$$|\boldsymbol{v_p}| = |\dot{\boldsymbol{r}}| = \sqrt{\dot{r}^2 + r^2 \dot{\varphi}^2}$$

$$\frac{\partial}{\partial r} |\boldsymbol{v_p}|^{\mathbf{2}} = \frac{\partial}{\partial r} (\dot{r}^2 + r^2 \dot{\varphi}^2)$$

$$2|\boldsymbol{v_p}| \frac{\partial}{\partial r} |\boldsymbol{v_p}| = 2\dot{r} \frac{\partial \dot{r}}{\partial r} + r^2 \frac{\partial}{\partial r} \dot{\varphi}^2 + \dot{\varphi}^2 \frac{\partial}{\partial r} r^2$$

$$2|\boldsymbol{v_p}| \frac{\partial}{\partial r} |\boldsymbol{v_p}| = -2\dot{r} \frac{\dot{r}}{r} + 2\dot{\varphi} r^2 \frac{\partial \dot{\varphi}}{\partial r} + 2r\dot{\varphi}^2$$

$$|\boldsymbol{v_p}| \frac{\partial}{\partial r} |\boldsymbol{v_p}| = -\frac{\dot{r}^2}{r} - 2\dot{\varphi}^2 r + r\dot{\varphi}^2$$

AD $$\frac{\partial}{\partial r} |\boldsymbol{v_p}| = -\frac{1}{|\boldsymbol{v_p}|} \left(\frac{\dot{r}^2}{r} + r\dot{\varphi}^2 \right)$$

$$\frac{\partial}{\partial \varphi} |\boldsymbol{v_p}|^{\mathbf{2}} = \frac{\partial}{\partial \varphi} (\dot{r}^2 + r^2 \dot{\varphi}^2)$$

$$2|\boldsymbol{v_p}| \frac{\partial}{\partial \varphi} |\boldsymbol{v_p}| = 2\dot{r} \frac{\partial \dot{r}}{\partial \varphi} + r^2 \frac{\partial}{\partial \varphi} \dot{\varphi}^2$$

$$2|\boldsymbol{v_p}| \frac{\partial}{\partial \varphi} |\boldsymbol{v_p}| = 2\dot{r} \frac{\partial \dot{r}}{\partial \varphi}$$

AE $$\frac{\partial}{\partial \varphi} |\boldsymbol{v_p}| = \frac{\dot{r}^2}{|\boldsymbol{v_p}|} \left\{ \frac{1}{\tan(\varphi - \varphi_{PH})} + \frac{\dot{r}}{r \dot{\varphi}} \right\}$$

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Sun quaternion on planet position:

$$\boldsymbol{q_S} = \sqrt{\frac{2Gm_S}{r}} + k_S(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}})\sqrt{\frac{2Gm_S}{r}} = \sqrt{\frac{2Gm_S}{r}}[1 + k_S(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}})]$$

Planet quaternion on Sun's position:

$$\boldsymbol{q_p} = \sqrt{\frac{2Gm_p}{r}} + k_p[\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times (-\hat{\boldsymbol{r}})]\sqrt{\frac{2Gm_p}{r}} = \sqrt{\frac{2Gm_p}{r}}\left[1 - k_p\left(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}\right)\right]$$

Relative quaternions on each position:

$$\boldsymbol{q_{RS}} = \sqrt{\frac{2Gm_{S0}}{r}}[1 + k_S(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}})] - \dot{\boldsymbol{r}}$$

or:

$$\boldsymbol{q_{RS}} = \sqrt{\frac{2Gm_{S0}}{r}}\left[1 + k_S(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) - \dot{\boldsymbol{r}}\sqrt{\frac{r}{2Gm_{S0}}}\right]$$

$$\boldsymbol{q_{Rp}} = \sqrt{\frac{2Gm_{p0}}{r}}\left[1 - k_p\left(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}\right)\right] + \sqrt{\frac{m_{p0}}{m_{S0}}}\dot{\boldsymbol{r}}$$

or:

$$\boldsymbol{q_{Rp}} = \sqrt{\frac{2Gm_{p0}}{r}}\left[1 - k_p\left(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}\right) + \dot{\boldsymbol{r}}\sqrt{\frac{r}{2Gm_{S0}}}\right]$$

Effective field multiplier:

$$\gamma_e = \left[1 + k_S(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) - \dot{\boldsymbol{r}}\sqrt{\frac{r}{2Gm_{S0}}}\right] \cdot \left[1 - k_p\left(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}\right) + \dot{\boldsymbol{r}}\sqrt{\frac{r}{2Gm_{S0}}}\right]$$

Or:

$$\gamma_{Re} = 1 - k_S k_p\left(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}\right) \cdot (\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) + \left[k_S(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} + k_p\left(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}\right) \cdot \dot{\boldsymbol{r}}\right]\sqrt{\frac{r}{2Gm_{S0}}} - \frac{r\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}}{2Gm_{S0}}$$

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$$\widehat{\boldsymbol{\omega}}_p = \left(\sin\theta_p \sin\theta \sin\varphi + \cos\theta_p \cos\theta\right)\hat{\boldsymbol{r}} + \left(\sin\theta_p \cos\theta \sin\varphi - \cos\theta_p \sin\theta\right)\widehat{\boldsymbol{\theta}} + \sin\theta_p \cos\varphi\, \widehat{\boldsymbol{\varphi}}$$

$$\widehat{\boldsymbol{\omega}}_S = [\sin\theta_S \sin\theta \cos(\varphi - \varphi_S) + \cos\theta_S \cos\theta]\hat{\boldsymbol{r}} + [\sin\theta_S \cos\theta \cos(\varphi - \varphi_S) - \cos\theta_S \sin\theta]\,\widehat{\boldsymbol{\theta}} - \sin\theta_S \sin(\varphi - \varphi_S)\,\widehat{\boldsymbol{\varphi}}$$

$$\boldsymbol{v_p} = \frac{d\boldsymbol{r}}{dt} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}$$

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$$(\hat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = \{-\sin\theta_S \sin(\varphi - \varphi_S)\hat{\boldsymbol{\theta}} - [\sin\theta_S \cos\theta \cos(\varphi - \varphi_S) - \cos\theta_S \sin\theta]\,\hat{\boldsymbol{\varphi}}\} \cdot (\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}})$$

$$(\hat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = -\sin\theta_S \sin(\varphi - \varphi_S) r\dot{\theta} - [\sin\theta_S \cos\theta \cos(\varphi - \varphi_S) - \cos\theta_S \sin\theta]\, r\dot{\varphi}\sin\theta$$

On the orbit plane:

$$(\hat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = r\dot{\varphi}\cos\theta_S$$

$$(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = [-\sin\theta_p \cos\varphi\,\hat{\boldsymbol{\theta}} - (\sin\theta_p \cos\theta \sin\varphi - \cos\theta_p \sin\theta)\hat{\boldsymbol{\varphi}}] \cdot (\dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}})$$

$$(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = -\sin\theta_p \cos\varphi\, r\dot{\theta} - (\sin\theta_p \cos\theta \sin\varphi - \cos\theta_p \sin\theta) r\dot{\varphi}\sin\theta$$

On the orbit plane:

$$(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = r\dot{\varphi}\cos\theta_p$$

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Partial derivatives:

$$\gamma_{Re} = 1 - k_S k_p (\hat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}) \cdot (\hat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) + [k_S (\hat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} + k_p (\hat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}}] \sqrt{\frac{r}{2Gm_{S0}}} - \frac{r\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}}{2Gm_{S0}}$$

AG $$-\frac{r\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}}{2Gm_{S0}} = -\frac{rv_p^2}{2Gm_{S0}}$$

AH $$(\hat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} \sqrt{\frac{r}{2Gm_{S0}}} = r\dot{\varphi}\cos\theta_p \sqrt{\frac{r}{2Gm_{S0}}}$$

AI $$(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} \sqrt{\frac{r}{2Gm_{S0}}} = r\dot{\varphi}\cos\theta_p \sqrt{\frac{r}{2Gm_{S0}}}$$

$$\frac{\partial}{\partial \mathrm{r}}\gamma_{Re} = \frac{\partial}{\partial \mathrm{r}}\left\{1 + [k_S (\hat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} + k_p (\hat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}}] \sqrt{\frac{r}{2Gm_{S0}}} - \frac{rv_p^2}{2Gm_{S0}}\right\}$$

$$\frac{\partial}{\partial \mathrm{r}}\gamma_{Re} = \frac{\partial}{\partial \mathrm{r}}\left\{[k_S \cos\theta_S + k_p \cos\theta_p] r\dot{\varphi} \sqrt{\frac{r}{2Gm_{S0}}} - \frac{rv_p^2}{2Gm_{S0}}\right\}$$

$$\frac{\partial}{\partial \mathrm{r}}\gamma_{Re} = (k_S \cos\theta_S + k_p \cos\theta_p)\left(\dot{\varphi}\sqrt{\frac{1}{2Gm_{S0}}}\frac{\partial \mathrm{r}^{\frac{3}{2}}}{\partial \mathrm{r}} + r\sqrt{\frac{r}{2Gm_{S0}}}\frac{\partial}{\partial \mathrm{r}}\dot{\varphi}\right) - \frac{v_p^2}{2Gm_{S0}}\frac{\partial \mathrm{r}}{\partial \mathrm{r}} - \frac{r}{2Gm_{S0}}\frac{\partial}{\partial \mathrm{r}}v_p^2$$

With $\frac{\partial\dot{\varphi}}{\partial r} = -2\frac{\dot{\varphi}}{r}$:

$$\frac{\partial}{\partial \mathrm{r}}\gamma_{Re} = (k_S\cos\theta_S + k_p\cos\theta_p)\left(\frac{3}{2}\dot{\varphi}\sqrt{\frac{r}{2Gm_{S0}}} - r\sqrt{\frac{r}{2Gm_{S0}}}\frac{2}{\mathrm{r}}\dot{\varphi}\right) - \frac{v_p^2}{2Gm_{S0}} - \frac{2rv_p}{2Gm_{S0}}\frac{\partial v_p}{\partial \mathrm{r}}$$

AN $$\frac{\partial}{\partial \mathrm{r}}\gamma_{Re} = -\frac{\dot{\varphi}}{2}(k_S\cos\theta_S + k_p\cos\theta_p)\sqrt{\frac{r}{2Gm_{S0}}} - \frac{rv_p}{2Gm_{S0}}\left(\frac{v_p}{r} + 2\frac{\partial v_p}{\partial \mathrm{r}}\right)$$

AK $$-\frac{\dot{\varphi}}{2}\cos\theta_S\sqrt{\frac{r}{2Gm_{S0}}}$$

AL $$-\frac{\dot{\varphi}}{2}\cos\theta_p\sqrt{\frac{r}{2Gm_{S0}}}$$

AM $$-\frac{rv_p}{2Gm_{S0}}\left(\frac{v_p}{r} + 2\frac{\partial v_p}{\partial \mathrm{r}}\right)$$

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$$\frac{\partial}{\partial\theta}\gamma_{ER} = \frac{\partial}{\partial\theta}\left\{1 - k_S k_p(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}})\cdot(\hat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}}) + \left[k_S(\hat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}})\cdot\dot{\boldsymbol{r}} + k_p(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}})\cdot\dot{\boldsymbol{r}}\right]\sqrt{\frac{r}{2Gm_{S0}}} + \frac{rv_p^2}{2Gm_{S0}}\right\}$$

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}})\cdot(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}})$$
$$= \frac{\partial}{\partial\theta}\{\sin\theta_S\sin(\varphi-\varphi_S)\sin\theta_p\cos\varphi + \sin\theta_p\sin\theta_S\cos^2\theta\cos(\varphi-\varphi_S)\sin\varphi$$
$$- [\sin\theta_p\cos\theta_S\sin\varphi + \sin\theta_S\cos\theta_p\cos(\varphi-\varphi_S)]\sin\theta\cos\theta + \cos\theta_p\sin^2\theta\cos\theta_S\}$$

On the orbit plane:

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}})\cdot(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}}) = -\frac{\partial}{\partial\theta}[\sin\theta_p\cos\theta_S\sin\varphi + \sin\theta_S\cos\theta_p\cos(\varphi-\varphi_S)]\sin\theta\cos\theta$$

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}})\cdot(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}}) = -0{,}5[\sin\theta_p\cos\theta_S\sin\varphi + \sin\theta_S\cos\theta_p\cos(\varphi-\varphi_S)]\frac{\partial}{\partial\theta}\sin 2\theta$$

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}})\cdot(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}}) = -[\sin\theta_p\cos\theta_S\sin\varphi + \sin\theta_S\cos\theta_p\cos(\varphi-\varphi_S)]\cos 2\theta$$

On the orbit plane ($\cos 2\theta = \cos\pi = -1$):

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}})\cdot(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}}) = \sin\theta_p\cos\theta_S\sin\varphi + \sin\theta_S\cos\theta_p\cos(\varphi-\varphi_S)$$

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$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}})\cdot\dot{\boldsymbol{r}} = \frac{\partial}{\partial\theta}[-\sin\theta_p\cos\varphi\, r\dot{\theta} - (\sin\theta_p\cos\theta\sin\varphi - \cos\theta_p\sin\theta)r\dot{\varphi}\sin\theta]$$

$\frac{\partial \dot{\varphi}}{\partial \theta} = 0$ and $\frac{\partial \dot{\theta}}{\partial \theta} = 0$ ile

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_p \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = -\sin\theta_p \cos\varphi\, r \frac{\partial\dot{\theta}}{\partial\theta} - r\dot{\varphi}\left[\left(\sin\theta_p \cos\theta \sin\varphi - \cos\theta_p \sin\theta\right)\frac{\partial \sin\theta}{\partial\theta} + \sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta_p \cos\theta \sin\varphi - \cos\theta_p \sin\theta\right)\right]$$

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_p \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = -r\dot{\varphi}\left[\sin\theta\left(\sin\theta_p \sin\varphi \frac{\partial\cos\theta}{\partial\theta} - \cos\theta_p \frac{\partial}{\partial\theta}\sin\theta\right)\right]$$

On the orbit plane:

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_p \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = r\dot{\varphi}\sin\theta_p \sin\varphi$$

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$$(\hat{\boldsymbol{\omega}}_S \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = -\sin\theta_S \sin(\varphi-\varphi_S) r\dot{\theta} - [\sin\theta_S \cos\theta \cos(\varphi-\varphi_S) - \cos\theta_S \sin\theta]\, r\dot{\varphi}\sin\theta$$

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_S \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = \frac{\partial}{\partial\theta}\{-\sin\theta_S \sin(\varphi-\varphi_S) r\dot{\theta} - [\sin\theta_S \cos\theta \cos(\varphi-\varphi_S) - \cos\theta_S \sin\theta]\, r\dot{\varphi}\sin\theta\}$$

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_S \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = -\sin\theta_S \sin(\varphi-\varphi_S)\, r\frac{\partial\dot{\theta}}{\partial\theta} - \frac{\partial}{\partial\theta}[\sin\theta_S \cos\theta \cos(\varphi-\varphi_S) - \cos\theta_S \sin\theta]\, r\dot{\varphi}\sin\theta$$

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_S \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = -r\dot{\varphi}\left\{\sin\theta \frac{\partial}{\partial\theta}[\sin\theta_S \cos\theta \cos(\varphi-\varphi_S) - \cos\theta_S \sin\theta] + [\sin\theta_S \cos\theta \cos(\varphi-\varphi_S) - \cos\theta_S \sin\theta]\frac{\partial}{\partial\theta}\sin\theta\right\}$$

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_S \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = -r\dot{\varphi}\left[\sin\theta_S \cos(\varphi-\varphi_S)\frac{\partial}{\partial\theta}\cos\theta - \cos\theta_S \frac{\partial}{\partial\theta}\sin\theta\right]$$

$$\frac{\partial}{\partial\theta}(\hat{\boldsymbol{\omega}}_S \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = r\dot{\varphi}\sin\theta_S \cos(\varphi-\varphi_S)$$

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Finaly:

AS

$$\frac{\partial}{\partial\theta}\gamma_{Re} = -k_S k_p[\sin\theta_S \cos\theta_p \cos(\varphi-\varphi_S) + \sin\theta_p \cos\theta_S \sin\varphi] + r\dot{\varphi}\sqrt{\frac{r}{2Gm_{S0}}}[k_S \sin\theta_S \cos(\varphi-\varphi_S) + k_p \sin\theta_p \sin\varphi]$$

AO

$$\sin\theta_S \cos\theta_p \cos(\varphi-\varphi_S)$$

AP $\sin\theta_p \cos\theta_S \sin\varphi$

AQ $r\dot{\varphi}\sqrt{\frac{r}{2Gm_{S0}}}\sin\theta_S\cos(\varphi-\varphi_S)$

AR $r\dot{\varphi}\sqrt{\frac{r}{2Gm_{S0}}}\sin\theta_p\sin\varphi$

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$$\gamma_{Re} = 1 - k_S k_p(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}})\cdot(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}}) + [k_S\cos\theta_S + k_p\cos\theta_p]r\dot{\varphi}\sqrt{\frac{r}{2Gm_{S0}}} - \frac{rv_p^2}{2Gm_{S0}}$$

$$\frac{\partial}{\partial\varphi}\gamma_{Re} = \frac{\partial}{\partial\varphi}\left\{-k_S k_p(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}})\cdot(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}}) - [k_S(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}})\cdot\dot{\boldsymbol{r}} - k_p(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}})\cdot\dot{\boldsymbol{r}}]\sqrt{\frac{r}{2Gm_{S0}}} - \frac{rv_p^2}{2Gm_{S0}}\right\}$$

With $\frac{\partial\dot{\varphi}}{\partial\varphi} = 0$;

$$\frac{\partial}{\partial\varphi}\gamma_{Re} = -\frac{r}{2Gm_{S0}}\frac{\partial}{\partial\varphi}v_p^2$$

$$\frac{\partial}{\partial\varphi}\gamma_{Re} = -\frac{rv_p}{Gm_{S0}}\frac{\partial v_p}{\partial\varphi}$$

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$$\frac{\partial}{\partial\varphi}(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}})\cdot(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}})$$

$$= \frac{\partial}{\partial\varphi}\{\sin\theta_S\sin(\varphi-\varphi_S)\sin\theta_p\cos\varphi + \sin\theta_p\sin\theta_S\cos^2\theta\cos(\varphi-\varphi_S)\sin\varphi$$
$$- [\sin\theta_p\cos\theta_S\sin\varphi + \sin\theta_S\cos\theta_p\cos(\varphi-\varphi_S)]\sin\theta\cos\theta + \cos\theta_p\sin^2\theta\cos\theta_S\}$$

On the orbital plane:

$$\frac{\partial}{\partial\varphi}(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}})\cdot(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}}) = \frac{\partial}{\partial\varphi}[\sin\theta_S\sin(\varphi-\varphi_S)\sin\theta_p\cos\varphi]$$

$$\frac{\partial}{\partial\varphi}(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}})\cdot(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}}) = \sin\theta_S\sin(\varphi-\varphi_S)\sin\theta_p\frac{\partial}{\partial\varphi}\cos\varphi + \sin\theta_S\sin\theta_p\cos\varphi\frac{\partial}{\partial\varphi}\sin(\varphi-\varphi_S)$$

$$\frac{\partial}{\partial\varphi}(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}})\cdot(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}}) = \sin\theta_S\sin\theta_p\cos\varphi\cos(\varphi-\varphi_S) - \sin\theta_S\sin(\varphi-\varphi_S)\sin\theta_p\sin\varphi$$

$$\frac{\partial}{\partial\varphi}(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}}\times\hat{\boldsymbol{r}})\cdot(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}}\times\hat{\boldsymbol{r}}) = \sin\theta_S\sin\theta_p[\cos(\varphi-\varphi_S)\cos\varphi - \sin\theta_S\sin(\varphi-\varphi_S)\sin\theta_p\sin\varphi]$$

$$\frac{\partial}{\partial\varphi}(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) \cdot (\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}) = \sin\theta_S \sin\theta_p \cos(2\varphi - \varphi_S)$$

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$$\frac{\partial}{\partial\varphi}(\widehat{\boldsymbol{\omega}}_{\boldsymbol{S}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = \frac{\partial}{\partial\varphi} r\dot{\varphi}\cos\theta_S = 0$$

$$\frac{\partial}{\partial\varphi}(\widehat{\boldsymbol{\omega}}_{\boldsymbol{p}} \times \hat{\boldsymbol{r}}) \cdot \dot{\boldsymbol{r}} = \frac{\partial}{\partial\varphi} r\dot{\varphi}\cos\theta_p = 0$$

Finally:

$$\frac{\partial}{\partial\varphi}\gamma_{Re} = -k_S k_p \sin\theta_S \sin\theta_p \cos(2\varphi - \varphi_S) + \frac{r v_p}{G m_{S0}} \frac{\partial v_p}{\partial\varphi}$$

AT $$\sin\theta_S \sin\theta_p \cos(2\varphi - \varphi_S)$$

AU $$\frac{r v_p}{G m_{S0}} \frac{\partial v_p}{\partial\varphi}$$

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Universal Motion Equation:

$$\boldsymbol{\nabla}\left(\frac{\gamma_{Re}}{r}\right) = 0$$

Gradient of a scalar field in Cartesian and spherical coordinates[3]:

$$\boldsymbol{\nabla}\Phi = \frac{\partial\Phi}{\partial x}\boldsymbol{i} + \frac{\partial\Phi}{\partial y}\boldsymbol{j} + \frac{\partial\Phi}{\partial z}\boldsymbol{k}$$

$$\boldsymbol{\nabla}\Phi = \frac{\partial\Phi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\widehat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}\widehat{\boldsymbol{\varphi}}$$

$$\boldsymbol{\nabla}\left(\frac{\gamma_{Re}}{r}\right) = \gamma_{Re}\boldsymbol{\nabla}\left(\frac{1}{r}\right) + \frac{1}{r}\boldsymbol{\nabla}\gamma_{Re}$$

In spherical coordinates:

$$\boldsymbol{\nabla}\left(\frac{\gamma_{Re}}{r}\right) = -\frac{\gamma_{Re}}{r^2}\hat{\boldsymbol{r}} + \frac{1}{r}\boldsymbol{\nabla}\gamma_{Re}$$

$$\boldsymbol{\nabla}\left(\frac{\gamma_{Re}}{r}\right) = -\frac{\gamma_{Re}}{r^2}\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{\partial\gamma_{Re}}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\gamma_{Re}}{\partial\theta}\widehat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\gamma_{Re}}{\partial\varphi}\widehat{\boldsymbol{\varphi}}\right)$$

$$\boldsymbol{\nabla}\left(\frac{\gamma_{Re}}{r}\right) = -\frac{\gamma_{Re}}{r^2}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\gamma_{Re}}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r^2}\frac{\partial\gamma_{Re}}{\partial\theta}\widehat{\boldsymbol{\theta}} + \frac{1}{r^2\sin\theta}\frac{\partial\gamma_{Re}}{\partial\varphi}\widehat{\boldsymbol{\varphi}}$$

$$\nabla\left(\frac{\gamma_{Re}}{r}\right)=\left(-\frac{\gamma_{Re}}{r^2}+\frac{1}{r}\frac{\partial\gamma_{Re}}{\partial r}\right)\hat{\boldsymbol{r}}+\frac{1}{r^2}\frac{\partial\gamma_{Re}}{\partial\theta}\hat{\boldsymbol{\theta}}+\frac{1}{r^2\sin\theta}\frac{\partial\gamma_{Re}}{\partial\varphi}\hat{\boldsymbol{\varphi}}$$

With universal motion equation in three axis we get:

$$-\frac{\gamma_{Re}}{r}+\frac{\partial\gamma_{Re}}{\partial r}=0$$

$$\frac{\partial\gamma_{Re}}{\partial\theta}=0$$

$$\frac{\partial\gamma_{Re}}{\partial\varphi}=0$$

Using γ_{Re} and partial derivatives the radial component:

$$\left(\gamma_{Re}-r\frac{\partial\gamma_{Re}}{\partial \mathrm{r}}\right)=\mathbf{0}$$

$$1-k_Sk_p\left[\sin\theta_S\sin\theta_p\sin(\varphi-\varphi_S)\cos\varphi+\cos\theta_p\cos\theta_S\right]+\left[k_S\cos\theta_S+k_p\cos\theta_p\right]r\dot{\varphi}\sqrt{\frac{r}{2Gm_{S0}}}$$
$$-\frac{rv_p^2}{2Gm_{S0}}-r\left[-\frac{\dot{\varphi}}{2}\left(k_S\cos\theta_S+k_p\cos\theta_p\right)\sqrt{\frac{r}{2Gm_{S0}}}-\frac{rv_p}{2Gm_{S0}}\left(\frac{v_p}{r}+2\frac{\partial v_p}{\partial \mathrm{r}}\right)\right]=0$$

$$1-k_Sk_p\left[\sin\theta_S\sin\theta_p\sin(\varphi-\varphi_S)\cos\varphi+\cos\theta_p\cos\theta_S\right]+\left(k_S\cos\theta_S+k_p\cos\theta_p\right)r\dot{\varphi}\sqrt{\frac{r}{2Gm_{S0}}}$$
$$-\frac{rv_p^2}{2Gm_{S0}}+\left(k_S\cos\theta_S+k_p\cos\theta_p\right)\frac{r\dot{\varphi}}{2}\sqrt{\frac{r}{2Gm_{S0}}}+\frac{rv_p^2}{2Gm_{S0}}+\frac{r^2v_p}{Gm_{S0}}\frac{\partial v_p}{\partial \mathrm{r}}=0$$

$$\left(\gamma_{Re}-r\frac{\partial\gamma_{Re}}{\partial \mathrm{r}}\right)=1-k_Sk_p\left[\sin\theta_S\sin\theta_p\sin(\varphi-\varphi_S)\cos\varphi+\cos\theta_p\cos\theta_S\right]$$
$$+\frac{3}{2}\left(k_S\cos\theta_S+k_p\cos\theta_p\right)r\dot{\varphi}\sqrt{\frac{r}{2Gm_{S0}}}+\frac{r^2v_p}{Gm_{S0}}\frac{\partial v_p}{\partial r}=0$$

Universal force equation for Sun and planet:

$$\boldsymbol{F}=Gm_Sm_p\nabla\left(\frac{\gamma_{Re}}{r}\right)$$

Or:

$$\boldsymbol{F}=Gm_Sm_p\left(-\frac{\gamma_{Re}}{r^2}+\frac{1}{r}\frac{\partial\gamma_{Re}}{\partial r}\right)\hat{\boldsymbol{r}}+\frac{1}{r^2}\frac{\partial\gamma_{Re}}{\partial\theta}\hat{\boldsymbol{\theta}}+\frac{1}{r^2\sin\theta}\frac{\partial\gamma_{Re}}{\partial\varphi}\hat{\boldsymbol{\varphi}}$$

$$F_r = -\frac{Gm_S m_p}{r^2}\left(\gamma_{Re} - r\frac{\partial \gamma_{Re}}{\partial r}\right) = 0$$

$$F_{\boldsymbol{\theta}} = \frac{Gm_S m_p}{r^2}\frac{\partial \gamma_{Re}}{\partial \theta} = 0$$

$$\boldsymbol{F} = \frac{Gm_S m_p}{r^2 \sin\theta}\frac{\partial \gamma_{Re}}{\partial \varphi} = 0$$

$$\left(\gamma_{Re} - r\frac{\partial \gamma_{Re}}{\partial \mathrm{r}}\right) = 1 - k_S k_p\left[\sin\theta_S \sin\theta_p \sin(\varphi - \varphi_S)\cos\varphi + \cos\theta_p \cos\theta_S\right]$$
$$+\frac{3}{2}(k_S \cos\theta_S + k_p \cos\theta_p) r\dot{\varphi}\sqrt{\frac{r}{2Gm_{S0}}} + \frac{r^2 v_p}{Gm_{S0}}\frac{\partial v_p}{\partial r} = 0$$

Hence in this radial multiplier:

AW	1	Newton force multiplier
AX	$-k_S k_p\left[\sin\theta_S \sin\theta_p \sin(\varphi - \varphi_S)\cos\varphi + \cos\theta_p \cos\theta_S\right]$	Rotational repulsive force multiplier
AY and AZ	$\frac{3}{2}(k_S \cos\theta_S + k_p \cos\theta_p) r\dot{\varphi}\sqrt{\frac{r}{2Gm_{S0}}}$	Responses to repulsive forces
BA	$\frac{r^2 v_p}{Gm_{S0}}\frac{\partial v_p}{\partial r}$	Response to Newton force

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[1] **Field Analysis and Electromagnetics** Masour Javid and Philip Marshall Brown, 1963 by theMcGraw-Hill Book Company, Inc. *Catalog Card Number 62-22199 page:46*

[2] **The Astronomical Almanac** fort the year 2013, ISBN 978-0-7077-41284 page: E8

[3] **Vector Analysis** and an Introduction to Tensor Analysis Murray R Spiegel Metric Edition Schaum's Outline Series 1974 *Chapter 4 and page 137*